The Yamabe Problem

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Recall the famous Uniformization Theorem, which says that

> Every closed Riemannian two-manifold is conformally equivalent to another with constant sectional curvature.

This is an affirmative answer to a natural question. Indeed, a conformal change of metric amounts to a choice of one smooth function (the conformal factor), which on a two manifold is being asked to satisfy one condition: that its scalar curvature, a single function, be constant. Indeed, recall that for a 2-mfd we have that $Rm = \frac{1}{4} S\, g \oslash g$, so that the scalar curvature determines the sectional curvature.

In higher dimensions, asking for a single conformal factor to control all of the sectional curvatures is highly overdetermined, since the full curvature tensor depends on more than just S: Recall that

$$\dim M = 3 : \quad Rm = Ric \oslash g - \frac{1}{4} S\, g \oslash g,$$

and in

$$\dim M \geq 4 : \quad Rm = W + \frac{1}{n-2} \mathring{Ric} \oslash g + \frac{1}{2n(n-1)} S\, g \oslash g$$

where W is the conformally invariant Weyl tensor.

Indeed, Rm has $\sim n^4$ component functions which we are attempting to influence with the one degree of freedom afforded by the conformal factor.

In higher dimensions, the natural result to ask for is therefore

The Yamabe Problem (1960): Let (M, g) be a closed Rm. mfd. of $\dim M = n \geq 3$. Can one find a metric $\tilde{g}$ conformal to g, which has constant scalar curvature?

Fortunately, the answer is yes! However, Yamabe made an error in his proof, and it took about 25 years for the problem to be resolved via the work of Trudinger, Aubin, and Schoen.

Here's how our exposition of this story will play out:

- **Part I:** Preliminaries
- **Part II:** The Outline of the Arguments
- **Part III:** The Yamabe Problem on the Sphere
- **Part IV:** Solving the Problem when $\lambda(M) < \lambda(S^n)$
- **Part V:** Reducing to the Case $\lambda(M) < \lambda(S^n)$

PART I: Preliminaries

Let (M,g) be a closed Rm. mfd of $\dim M = n \geq 3$, and suppose that $\tilde{g} = e^{2f} g$ is a metric conformal to g (here, $f \in C^\infty(M)$). Here and throughout, quantities on $(M,\tilde{g})$ will be written with $\sim$'s, and corresponding quantities on (M,g) without. We begin by observing how key geometric quantities evolve under conformal changes:

Metric: $g \longmapsto \tilde{g} = e^{2f} g$ for $f \in C^\infty(M)$ **Volume:** $dvol \mapsto e^{nf} dvol_g$

Christoffel Symbols: $\Gamma_{ij}^k = \frac{1}{2} g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij})$

$$\longmapsto \tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \tfrac{1}{2}(\delta_j^k \partial_i f + \delta_i^k \partial_j f - g_{ij} g^{kl} \partial_l f)$$

Riemann Curvature: Note: Our Laplacian Convention is $\Delta f = \mathrm{div}(\mathrm{grad} f) = \mathrm{tr}\,\nabla^2 f$.

$$Rm \longmapsto \tilde{Rm} = e^{2f}\left\{ Rm - (\nabla^2 f) \owedge g + (df \otimes df) \owedge g - \tfrac{1}{2}|df|_g^2\, g \owedge g \right\}$$

Ricci Curvature:

$$Ric \longmapsto \tilde{Ric} = Ric - (n-2)(\nabla^2 f) + (n-2)(df \otimes df) - \left\{ \Delta f + (n-2)|df|_g^2 \right\} g$$

Scalar Curvature:

$$S \longmapsto \tilde{S} = e^{-2f}\left\{ S - 2(n-1)\Delta f - (n-1)(n-2)|df|_g^2 \right\}$$

Schouten Tensor: $A = \frac{1}{n-2}\left\{ Ric - \frac{S}{2(n-1)} g \right\}$

$$\longmapsto \tilde{A} = A - \nabla^2 f + df \otimes df - \tfrac{1}{2}|df|_g^2\, g$$

Weyl Tensor: $W = Rm - A \owedge g \longmapsto \tilde{W} = e^{2f} W$

Notice! This last formula shows that W is conformally invariant! Some facts about the Weyl tensor that will be important later on are that

- If $n=3$, $W \equiv 0$
- If $n \geq 4$, $W(p) = 0 \rightarrow (M,g)$ is locally conformally flat at p.

Now, consider the formula above for scalar curvature. We can simplify it if we write $e^{2f} = \varphi^{p-2}$, where from here on $p := 2n/(n-2)$. Then

$$\tilde{S} = \varphi^{1-p}\left\{ -c_n \Delta\varphi + S\varphi \right\} \equiv \varphi^{1-p} \Box \varphi$$

where $c_n = \frac{4(n-1)}{n-2}$ and $\Box := -c_n \Delta + S$ is the conformal Laplacian.

From this, we see that solving the Yamabe Problem is equivalent to finding a smooth, positive solution to the nonlinear eigenvalue problem

(The Yamabe Equation) $\qquad \Box \varphi = \lambda \varphi^{p-1} \qquad (\star)$

for some $\lambda \in \mathbb{R}$, which will be the (constant!) scalar curvature of the new metric $\tilde{g} = \varphi^{p-2} g$.

As an aside, $\Box$ is conformally invariant in the sense that under a conformal change $\tilde{g} = e^{2f} g$,

$$\tilde{\Box} u = e^{-\frac{n+2}{2} f} \Box (e^{\frac{n-2}{2}} u) \qquad \forall u \in C^\infty(M).$$

If instead we write $\tilde{g} = \varphi^{p-2} g$, then the conformal invariance takes the form

$$\tilde{\Box}(\varphi^{-1} u) = \varphi^{1-p} \Box u.$$

PART II: The Outline of the Arguments:

Yamabe's first observation was that ⊛ is the Euler-Lagrange equation for the Q-functional

$$Q(\varphi) \equiv Q_g(\varphi) := \frac{\int c_n |\nabla \varphi|^2 + S \varphi^2 \, d\mathrm{vol}_g}{\|\varphi\|_p^2} = \frac{\int \tilde{S} \, d\mathrm{vol}_{\tilde{g}}}{\mathrm{vol}_{\tilde{g}}(M)^{2/p}} =: Q(\tilde{g})$$

with $\tilde{g} = \varphi^{2-p} g$ varying over the conformal class of g. In the third equality, we use $\tilde{S} = \varphi^{1-p} \Box \varphi = \{-c_n \varphi \Delta \varphi + \varphi^2\} \varphi^{-p}$ and $d\mathrm{vol}_{\tilde{g}} = (\varphi^{p-2})^{\frac{n}{2}} d\mathrm{vol}_g = \varphi^p d\mathrm{vol}$.

Claim: Let $\varphi \in W^{1,2}(M)$ be a critical point of Q. Then there is some $\lambda \in \mathbb{R}$ s.t. φ satisfies ⊛. Moreover, if φ is a minimizer and $\|u\|_p = 1$, then

$$\lambda = \lambda(M) := \inf\{Q(\varphi) : \varphi \in W^{1,2}(M)\}$$

(since Q is cts on $W^{1,2}$, $Q(|\varphi|) = Q(\varphi)$, and density)

$$:= \inf\{Q(\varphi) : \varphi \in C^\infty(M, \mathbb{R}^{>0})\}$$

$$= \inf\{Q(\tilde{g}) : \tilde{g} \text{ is conformal to } g\}.$$

Remark: $\lambda(M)$ is called the Yamabe Constant, and is evidently an invariant of the conformal class of (M, g) since $Q_g(\varphi\psi) = Q_{\varphi^{p-2}g}(\psi)$.

Proof of Claim: Let $\xi \in C_c^\infty(M)$. Then if $\varphi \in W^{1,2}(M)$ is a critical point of Q,

$$0 = \frac{d}{dt}\Big|_{t=0} Q(\varphi + t\xi) = \frac{d}{dt}\Big|_{t=0} \frac{\int_M c_n |\nabla \varphi + t \nabla \xi|^2 + S(\varphi + t\xi)^2 \, d\mathrm{vol}_g}{\|\varphi + t\xi\|_p^2}$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{\int c_n |\nabla \varphi|^2 + 2c_n t \langle \nabla\varphi, \nabla\xi\rangle + c_n t^2 |\nabla \xi|^2 + S\varphi^2 + 2tS\varphi\xi + t^2 S \xi^2 \, d\mathrm{vol}_g}{\|\varphi + t\xi\|_p^2}$$

$$= \frac{\|\varphi\|_p^2 \left\{2\int c_n \langle \nabla\varphi, \nabla\xi\rangle + S\varphi\xi \, d\mathrm{vol}_g\right\} - \left\{\int c_n |\nabla\varphi|^2 + S\varphi^2 d\mathrm{vol}_g\right\} 2\|\varphi\|_p^{2-p} \int \xi \varphi^{p-1}}{\|\varphi\|_p^4}$$

$$\longrightarrow \quad 0 = \frac{2}{\|\varphi\|_p^2} \int \xi \left(\square \varphi - Q(\varphi) \frac{\varphi^{p-1}}{\|\varphi\|_p^{p-2}} \right) d\mathrm{vol}_g$$

By arbitrariness of ξ, we conclude that φ solves $\star$ with $\lambda = \dfrac{Q(\varphi)}{\|\varphi\|_p^{p-2}}$.

If now φ is a minimizer with $\|\varphi\|_p = 1$, then $\lambda = Q(\varphi) = \lambda(M)$. ∎

We should note that this variational problem yields a finite $\lambda(M)$, since by the Hölder and Sobolev Inequalities

$$\frac{\int c_n |\nabla \varphi|^2 + S\varphi^2 \, d\mathrm{vol}_g}{\|\varphi\|_p^2} \geqslant C - \|S\|_{n/2} > -\infty$$

while $\lambda(M) < Q(1) = \dfrac{\|S\|_1}{\mathrm{vol}(M)^{2/2^*}} < \infty$.

So, to solve the Yamabe Problem, it suffices to find a minimizer of $\lambda(M)$. However, we can't just use the direct method because, amusingly, the exponent $p = 2^* = \frac{2n}{n-2}$ that appears in Yamabe's Equation is exactly the critical Sobolev exponent where the compact embedding $W^{1,2} \hookrightarrow L^p$ fails.

As we will see later, Yamabe's approach was to study the sub-critical problem with exponent $q < p = 2^*$, which is solvable by the direct method. The hope is then that the subcritical solutions converge to a solution of the Yamabe Equation with critical exponent. Yamabe had claimed a proof of this in 1960, but in 1968 Trudinger discovered a false claim – Yamabe had asserted the validity of uniform $C^{2,\alpha}$ estimates for the sequence of subcritical solutions, but this even fails on $(\mathbb{S}^n, g_{rd})$! It would take until 1984 for Trudinger, Aubin, Schoen, and others to rectify the argument, via the following results:

<u>Theorem</u>: (Yamabe, Trudinger, Aubin) If $\lambda(M) < \lambda(\mathbb{S}^n)$, then a positive, smooth minimizer of $\lambda(M)$ exists, solving the Yamabe Problem on M.

<u>The Idea</u>: Strict inequality gives us room to account for error terms as a result of the lack of compactness.

<u>Theorem</u>: (Aubin) If (M,g) has $\dim M = n \geqslant 6$, and if M is not locally conformally flat at some $p \in M$, then $\lambda(M) < \lambda(\mathbb{S}^n)$.

<u>The Idea</u>: By using test functions inspired by the resolution of the problem on $\mathbb{S}^n$, one can show directly that $\lambda(M) \leq Q(\varphi) < \lambda(\mathbb{S}^n)$.

<u>Theorem</u>: (Schoen) If (M,g) has $\dim M = n \in \{3,4,5\}$, or if M is locally conformally flat at some point, then $\lambda(M) < \lambda(\mathbb{S}^n)$.
(maybe with $\leq$ and rigidity with sphere)

<u>The Idea</u>: The remaining cases can't be tackled by local estimates, but Schoen discovered how to build the desired test function from the Green's function of $\square$. Fascinatingly, his proof relies on the PMT from mathematical relativity.

Part III: The Yamabe Problem on the Sphere, and the Sharp Sobolev Inequality

The case $M = S^n$ is interesting because we can not only give an explicit solution to the variational problem, but also because it is central to understanding the problem for general M. Even further, it has deep ties to the Sobolev Inequality, making the Yamabe Problem on S^n highly relevant to analysis at large!

In this section, we will see how the problem is resolved on S^n: the standard round metric minimizes the Q-funtional, and its conformal metrics are the only metrics on S^n with constant scalar curvature. We will in particular see how the Yamabe Problem on S^n is equivalent to the problem of determining the optimal constant for the Sobolev Inequality on $\mathbb{R}^n$. This gives two possible methods for solving the Yamabe Problem on S^n: either directly or by finding the sharp Sobolev constant. Lastly, we show that $\lambda(S^n)$ provides an upper bound for $\lambda(M)$ with M any compact Rm. mfd. of $\dim M \geq 3$.

Stereographic Projection:

Consider $S^n \hookrightarrow \mathbb{R}^{n+1}$ with its round metric g_0 induced by g_{std} on $\mathbb{R}^{n+1}$. Let $N = (0, \dots, 1)$ be the north pole of S^n, and recall that the mapping $\Psi : S^n \setminus N \to \mathbb{R}^n$ given by

$$(s^1, \dots, s^n, \xi) \longmapsto (x^1, \dots, x^n) \quad , \quad x^i = \frac{s^i}{1-\xi}$$

is a conformal diffeomorphism, with $(\Psi^{-1})^* g_0 = 4(1+|x|^2)^{-2} g_{std}$. For notational convenience, let $\rho = \Psi^{-1} : \mathbb{R}^n \to S^n \setminus \{N\}$.

We can also write this as $\rho^* g_0 = 4 u_1^{p-2} g_{std}$, where for $\alpha > 0$

$$u_\alpha(x) := \left(\frac{\alpha^2 + |x|^2}{\alpha} \right)^{\frac{2-n}{2}}$$

and as always, $p = 2^* = \frac{2n}{n-2}$. The point of introducing u_α is to describe how g_0 changes under conformal diffeomorphisms of the sphere. Indeed, the group of conformal diffeomorphisms of S^n is generated by the rotations in $O(n+1)$, and maps of the form $\Psi^{-1} \tau_v \Psi$, $\Psi^{-1} \delta_\alpha \Psi$, with $\tau_v, \delta_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ translation by $v \in \mathbb{R}^n$ and dilation by $\alpha > 0$, respectively. We have, in particular, that

$$(\rho \circ \delta_\alpha)^* g_0 = \delta_\alpha^* \cdot \rho^* g_0 = 4 u_\alpha^{p-2} g_{std}.$$

There are now two ways in which we may proceed, so let's explore both. In short, the S^n Yamabe Problem is equivalent to the problem of sharpening the Sobolev Inequality on $\mathbb{R}^n$. One can thus proceed by solving one problem or the other.

Route 1 to Solving the Yamabe Problem on S^n: (The Talenti-Aubin Approach)

Recall the famous Sobolev Inequality on $\mathbb{R}^n$: $\forall \varphi \in \underset{W^{1,2}(\mathbb{R}^n)}{C_c^1(\mathbb{R}^n)}$, $1 \le p < n$, $p^* = \frac{np}{n-p}$,

$$\|\varphi\|_{p^*} \le C_n \|\nabla \varphi\|_p .$$

It turns out that the conformal invariance of $\lambda(S^n)$ makes the resolution of the Yamabe Problem on S^n equivalent to the task of determining the sharp coefficient C_n in this Sobolev Inequality.

To see this, let $\varphi \in C^\infty(S^n)$, and define $\bar{\varphi} := u_1 \rho^* \varphi$. Then recalling the facts that

- $\rho : (S^n \setminus \{N\}, g_0) \to (\mathbb{R}^n, 4u_1^{p-2} dx)$ is an isometry;
- $\rho^*(\nabla^{\rho^* g_0} \rho^* \varphi) = \nabla^{g_0} \varphi$
- $Q_g(\lambda \varphi) = Q_g(\varphi)$ for every $\lambda \in \mathbb{R} \setminus \{0\}$
- $Q_g(\sigma \varphi) = Q_{\sigma^{p-2} g}(\varphi) \quad \forall \sigma \in C^\infty(M : \mathbb{R}^{>0})$

we obtain

$$\begin{aligned}
Q_{g_{std}}(\bar{\varphi}) = Q_{g_{std}}(u_1 \rho^* \varphi) &= Q_{g_{std}}(4^{\frac{1}{p-2}} u_1 \rho^* \varphi) \\
&= Q_{4 u_1^{p-2} g_{std}}(\rho^* \varphi) \\
&= Q_{\rho^* g_0}(\rho^* \varphi) \\
&= Q_{g_0}(\varphi)
\end{aligned}$$

To prove ●, we compute directly from the definition of Q and use the equation $\Box \varphi = \bar{S} \varphi^{p-1}$. To prove $=$ we again compute directly from Q_{g_0}, using ● and the fact that S is isometry invariant to compute that

$$Q_{g_0}(\varphi) = \frac{\int_{S^n \setminus \{N\}} c_n |\nabla \varphi|^2_{g_0} + S \varphi^2 \, d\mathrm{vol}_{g_0}}{\left(\int_{S^n \setminus \{N\}} |\varphi|^p \, d\mathrm{vol}_{g_0} \right)^{2/p}}$$

$$= \frac{\int_{\mathbb{R}^n} c_n |\nabla^{\rho^* g_0} \rho^* \varphi|^2_{\rho^* g_0} + \rho^* S_{\rho^* g_0} (\rho^* \varphi)^2 \, d\mathrm{vol}_{\rho^* g_0}}{\left(\int_{\mathbb{R}^n} |\rho^* \varphi|^p_{\rho^* g_0} \, d\mathrm{vol}_{\rho^* g_0} \right)^{2/p}}$$

$$= Q_{\rho^* g_0}(\rho^* \varphi) .$$

The upshot is that $Q_{g_{std}}(\overline{\varphi})$ has a simpler form, as the scalar curvature of $(\mathbb{R}^n, g_{std})$ vanishes. Thus,

$$\lambda(S^n) = \inf_\varphi Q_{S^n, g_0}(\varphi) = \inf_{\varphi \in C^\infty(S^n)} Q_{\mathbb{R}^n, g_{std}}(\overline{\varphi})$$

$$= \inf_{\varphi \in C^\infty(S^n)} \frac{\int_{\mathbb{R}^n} c_n |\nabla \overline{\varphi}|^2 dx}{\left(\int_{\mathbb{R}^n} |\overline{\varphi}|^p dx\right)^{2/p}}$$

By approximating $\overline{\varphi}$ with cutoff functions, it follows that

$$\lambda(S^n) = \inf_{\varphi \in C_0^\infty(\mathbb{R}^n)} \frac{c_n \|\nabla \varphi\|_2^2}{\|\varphi\|_p^2}$$

Theorem: (Talenti, Aubin): Let $n \geq 3$, and

$$\sigma_n^2 := \inf \left\{ \frac{\|\nabla u\|_2^2}{\|u\|_p^2} : u \in W^{1,2}(\mathbb{R}^n) \right\}.$$

Then $\sigma_n^2 = c_n^{-1} \cdot n(n-1) \omega_n^{2/n}$, and minimizers are exactly the constant multiples and translates of u_α as defined above.

Thus, the sharp Sobolev Inequality on $\mathbb{R}^n$ is

$$\|u\|_p \leq \frac{1}{\sigma_n} \|\nabla u\|_2 = \frac{c_n}{[n(n-1)]^{1/2} \omega_n^{1/n}} \|\nabla u\|_2 \qquad \forall u \in W^{1,2}(\mathbb{R}^n).$$

So, Talenti and Aubin thus solved the Yamabe Problem on the sphere, and gave an explicit value for $\lambda(S^n)$. Their proofs (independently discovered but essentially similar) consist mostly of technical GMT.

Corollary: If (M^n, g) is any closed Rm. mfd with $n \geq 3$, then $\lambda(M) \leq \lambda(S^n)$.

This is obtain by testing Q_g with the u_α above, localized to normal balls.

Part IV: Resolving the Yamabe Problem when $\lambda(M) < \lambda(S^n)$.

This part represents the most analytic side of the problem, and just as in the last part, there are multiple paths by which we may proceed. We'll outline both, which seek to prove the following:

Theorem: (Yamabe, Trudinger, Aubin)

Suppose $\lambda(M) < \lambda(S^n)$. Then a minimizer of $\lambda(M)$ exists, thus solving the Yamabe Problem on M.

The intuition here is that although the embedding $W^{1,2}(M) \hookrightarrow L^{2^*}(M)$ is not compact, a minimizing sequence which fails to converge to a minimizer would have to concentrate, or "bubble", at some point of M, and this would add a $\lambda(S^n)$ to the functional. Since $\lambda(M) < \lambda(S^n)$, this sort of concentration shouldn't be able to occur, and we can hope for convergence to a minimizer.

The first approach we'll outline is due to Lions' in 1984, as it beautifully exhibits the bubbling phenomenon. In fact, it says generally that a bounded sequence in $L^{2^*}(M)$ which doesn't converge strongly must concentrate at countably many points, and that the amount of concentration at each point can be controlled via a Sobolev-type inequality for measures. We'll use the Sharp Sobolev Inequality on $\mathbb{R}^n$ to obtain this control. Thus, we conclude that even though we lack compactness, we still have a pretty precise understanding of how badly compactness fails, and can use the strict inequality in $\lambda(M) < \lambda(S^n)$ to absorb the effects of the failure.

The second approach we'll detail is closer to Yamabe's original approach, and is due to Yamabe and completed by Trudinger and Aubin. The idea here is very interesting from a PDE perspective, and is based on the idea that the subcritical equations

$$\Box \varphi = \lambda_s \varphi^{s-1} \qquad (2 \leq s < p = 2^*)$$

associated to the perturbed functionals

$$Q^s(\varphi) = E(\varphi) / \|\varphi\|_s^2$$

are easy to solve (i.e., positive, smooth solutions φ_s with $\lambda_s = \inf_{C^\infty(M)} Q^s(\varphi)$ always exist). The difficulty, and the site of Yamabe's error, is in showing that these subcritical solutions converge to a solution of the critical equation with $s = p = 2^*$. He had claimed the validity of a uniform $C^{2,\alpha}$ estimate for the φ_s, in the hope of applying Arzela-Ascoli to obtain a limit. However, such a uniform estimate is false, in particular on S^n! Nonetheless, when $\lambda(M) < \lambda(S^n)$, these estimates do hold, as there is space to allow for the error terms.

The First Approach: Lions' Concentration-Compactness Lemma

Lemma: (Lions)

Let $\{u_k\} \subseteq W^{1,2}(M)$ be uniformly bounded, so that $u_k \rightharpoonup u \in W^{1,2}(M)$. Up to subsequences,

$$\mu_k := |\nabla u_k|^2 \, dvol_g \longrightarrow \mu$$
$$\nu_k := |u_k|^{2^*} dvol_g \longrightarrow \nu.$$

Moreover,

$$\mu \geq |\nabla u|^2 \, dvol_g + \sigma_n^2 \, \Sigma_J \alpha_j^{2/2^*} \delta_{p_j}$$
$$\nu = |u|^{2^*} dvol_g + \Sigma_J \alpha_j \delta_{p_j}$$

where J is countable and $p_j \in M$, $\alpha_j \in (0, \infty)$.

Before presenting Lions' proof of the above, let's see how it helps us prove the main theorem of this section:

Proof of the Theorem:

Let $\{\varphi_n\} \subset W^{1,2}(M)$ be a minimizing sequence for $\lambda(M)$, and WLOG take $\|\varphi_k\|_{2^*} = 1$. By the Sobolev Embedding Theorem, $W^{1,2}(M) \subset\subset L^2(M)$, so up to a subsequence $\varphi_n \to \varphi \in L^2(M)$, while $\varphi_n \overset{*}{\rightharpoonup} \varphi$ in both $W^{1,2}(M)$ and $L^{2^*}(M)$ by Banach-Alaoglu.

In particular, we know that $\|\varphi\|_{2^*} \leq \liminf \|\varphi_k\|_{2^*} = 1$, so $\exists t \in [0,1]$ so that $\|\varphi\|_{2^*}^{2^*} = t$. We aim to show that $t = 1$, because then we know that $\varphi_k \to \varphi$ in $L^{2^*}(M)$, and this will allow us to conclude that φ is a minimizer of $\lambda(M)$ in $W^{1,2}(M)$.

Indeed, φ minimizes $\lambda(M)$ iff it minimizes

$$E(\varphi) = \int c_n |\nabla \varphi|^2 + S\varphi^2 \, dvol_g - \lambda(M) \|\varphi\|_{2^*}^2$$

on $W^{1,2}(M)$ — just re-write this as $E(\varphi) = \{Q(\varphi) - \lambda(M)\} \|\varphi\|_{2^*}^2$. Evidently, our infimizing sequence $\{\varphi_n\}$ for Q is an infimizing sequence for E, and since

$$\begin{aligned} E(\varphi) &= \int c_n |\nabla\varphi|^2 + S\varphi^2 \, dvol_g - \lambda(M) \\ &\leq \liminf \left\{ \int c_n |\nabla \varphi_k|^2 + S\varphi_k^2 \, dvol_g - \lambda(M) \|\varphi_k\|_{2^*}^2 \right\} \\ &= \liminf E(\varphi_k) \end{aligned}$$

we see that $\varphi \in W^{1,2}(M)$ is a minimizer of E, and so also of Q. Here we notice how crucial it is that we have strong convergence in L^{2^*} for φ_k. If the limit φ were to lose mass, then we could lose lower semicontinuity of E, and wouldn't be able to conclude that φ minimizes.

So, let's turn to showing that $t = \|\varphi\|_{2^*}^{2^*} = 1$:

By Lions' Lemma,

$$\lambda(M) = \lim Q(\varphi_k) = \lim \int c_n |\nabla \varphi_k|^2 + S\varphi_k^2 \, dvol_g$$

$$\geq \int c_n |\nabla\varphi|^2 + S\varphi^2 \, dvol_g + c_n \sigma_n^2 \Sigma_J \alpha_j^{2/2^*}$$

$(\lambda(S^n) = c_n \sigma_n^2)$

$$= Q(\varphi) t^{2/2^*} + \lambda(S^n) \Sigma_J \alpha_j^{2/2^*}$$

$$\geq \lambda(M) t^{2/2^*} + \lambda(S^n) \Sigma_J \alpha_j^{2/2^*}$$

$$= \lambda(M) t^{2/2^*} + \lambda(S^n)(1-t)^{2/2^*} \Sigma_J \left(\frac{\alpha_j}{1-t}\right)^{2/2^*}$$

(since $2/2^* \in (0,1]$)

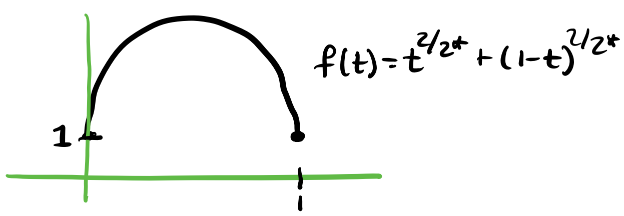
$$\geq \lambda(M) t^{2/2^*} + \lambda(S^n)(1-t)^{2/2^*} \left(\Sigma_J \frac{\alpha_j}{1-t}\right)^{2/2^*}$$

$$= \lambda(M) t^{2/2^*} + \lambda(S^n)(1-t)^{2/2^*}$$

since $1 \equiv \lim v_k(M) = v(M) = \|\varphi\|_{2^*}^{2^*} + \Sigma_J \alpha_j = t + \Sigma_J \alpha_j$. Applying our assumption that $\lambda(S^n) > \lambda(M)$, we obtain

$$\lambda(M) \geq t^{2/2^*} \lambda(M) + \lambda(S^n)(1-t)^{2/2^*} \geq \lambda(M)\{t^{2/2^*} + (1-t)^{2/2^*}\} \geq \lambda(M).$$

Equality $\geq$ tells us that $t \in \{0,1\}$. If $t=0$, $\geq$ would be strict. Thus $t=1$.



Now, we have in hand a minimizer $\varphi \in W^{1,2}(M)$ of Q. By the standard repertoire of elliptic regularity, which we'll see when we consider the second approach, it follows that φ is smooth and positive, thus solving the Yamabe Problem when $\lambda(M) < \lambda(S^n)$.

With this in hand, lets now return to Lions' Concentration-Compactness Lemma, and show why it holds.

Proof of Lions' Lemma: The proof of the result on M follows from the corresponding result on a bounded subset of $\mathbb{R}^n$, using normal coordinates and a partition of unity. Thus, it's most instructive to focus on the proof of the result in $\mathbb{R}^n$.

Set $v_k := (u_k - u)$, $\omega_k := (|u_k|^{2^*} - |u|^{2^*})dx$, $\tilde{\mu}_k := |\nabla v_k|^2 dx$. By assumption, $v_k \to 0$ in $W^{1,2}$ and L^{2^*}. By the uniform $W^{1,2}$ bounds on u_k, u, the compactness theorem for Radon measures yields Radon measures ω and $\tilde{\mu}$ so that $\omega_k \to \omega$ and $\tilde{\mu}_k \to \tilde{\mu}$.

It can be shown that $\omega_k = |v_k|^{2^*}dx + o(1) = |u_k - u|^{2^*}dx + o(1)$.

Fix $\xi \in C_c^\infty(\mathbb{R}^n)$, and use the Sobolev Inequality to obtain

$$\begin{aligned}\int \xi^{2^*} d\omega = \lim \int (\xi |v_k|)^{2^*} dx &\le \liminf \frac{1}{\sigma_n^{2^*}} \left(\int |\nabla(\xi v_k)|^2 dx\right)^{2^*/2} \\ &\le \liminf \frac{1}{\sigma_n^{2^*}} \left(\int \xi^2 |\nabla v_k|^2 dx\right)^{2^*/2} \\ &= \frac{1}{\sigma_n^{2^*}} \left(\int \xi^2 d\tilde{\mu}\right)^{2^*/2}.\end{aligned}$$

Note that this computation technically is true up to a subsequence of the v_k on which the compact embedding $W^{1,2} \subset\subset L^2$ ensures $v_k \to 0$, yielding

$$\|\nabla(\xi v_k)\|_2 \le \|\xi \nabla v_k\|_2 + \|v_k \nabla \xi\|_2$$

$$\longrightarrow \liminf \|\nabla(\xi v_k)\|_2 \le \liminf \|\xi \nabla v_k\|_2 = \|\xi\|_{L^2(\mathbb{R}^n, \tilde{\mu})}.$$

This establishes a "reverse Hölder Inequality" for ω and $\tilde{\mu}$:

$$\sigma_n \|\xi\|_{L^{2^*}(\mathbb{R}^n, \omega)} \le \|\xi\|_{L^2(\mathbb{R}^n, \tilde{\mu})} \qquad \forall \xi \in C_c^\infty(\mathbb{R}^n).$$

We apply this to a sequence of $\xi_k \in C_c^\infty(\mathbb{R}^n)$ approximating χ_Ω for an open $\Omega \subseteq \mathbb{R}^n$, which yields

$$\sigma_n^2 \omega(\Omega)^{2/2^*} \le \tilde{\mu}(\Omega).$$

The non-linearity of this control is what forces ω to be supported on a countable set of atoms. Indeed, $\tilde{\mu}$ is finite, and so can have at most countably many atoms $\{p_j\}$. If $x \in \mathbb{R}^n \setminus \{p_j\}$, then we can find an open $\Omega \ni x$ with $\tilde{\mu}(\Omega) \le \sigma_n^2$, and so

$$1 \ge \sigma_n^{-2} \tilde{\mu}(\Omega) \ge \omega(\Omega)^{2/2^*} \ge \omega(\Omega)$$

Thus, $\omega << \tilde{\mu}$ on $\mathbb{R}^n \setminus \{p_j\}$, and the Lebesgue-Besicovitch Theorem tells us that $D_{\tilde{\mu}}\omega \equiv 0$ a.e.: At any $x \in \mathbb{R}^n \setminus \{p_j\}$

$$D_{\tilde{\mu}}\omega(x) = \lim_{r \to 0} \frac{\omega(B_r(x))}{\tilde{\mu}(B_r(x))} \le \liminf_{r \to 0} \sigma_n^{-2^*} \mu(B_r(x))^{2^*/2 - 1} = 0$$

So, $\omega = (D_{\tilde{\mu}}\omega)\tilde{\mu} + \omega^s_{\tilde{\mu}} = \omega^s_{\tilde{\mu}}$, where the singular part $\omega^s_{\tilde{\mu}}$ is supported on the atomic points $\{p_j\}$. Thus,

$$\omega = \sum_J \alpha_j \delta_{p_j}$$

which proves that, as desired,

$$\nu_n := |u_n|^{2^*} dx \longrightarrow |u|^{2^*} dx + \sum_J \alpha_j \delta_{p_j} .$$

To prove the remaining statement, fix one of the p_j and apply our reverse Hölder Inequality above with $\xi_r \in C_c^\infty(\mathbb{R}^n)$ satisfying $\xi_r(p_j) = 1$, $\operatorname{spt}\xi_r \subseteq \overline{B_r}(p_j)$ as $r \to 0$:

$$\begin{aligned}\sigma_n^2 \alpha_j^{2/2^*} = \sigma_n^2 \omega(p_j)^{2/2^*} &= \lim_{r \to 0} \sigma_n^2 \left(\int \xi_r^{2^*} d\omega \right)^{2/2^*} \\ &\le \liminf_{r \to 0} \int \xi_r^2 \, d\tilde{\mu} \\ &= \tilde{\mu}(p_j).\end{aligned}$$

Thus,

$$\tilde{\mu} \ge \sigma_n^2 \sum_J \alpha_j^{2/2^*} \delta_{p_j}$$

Now, observe that

$$\begin{aligned}\tilde{\mu}_n = |\nabla u_n - \nabla u|^2 dx &= \left(|\nabla u_n|^2 + |\nabla u|^2 - 2\langle \nabla u_n, \nabla u \rangle \right) dx \\ &= \mu_n + \left(|\nabla u|^2 - 2\langle \nabla u_n, \nabla u \rangle \right) dx,\end{aligned}$$

so by uniqueness of weak limits

$$\mu - |\nabla u|^2 dx = \tilde{\mu}$$

and hence $$\mu \ge |\nabla u|^2 dx + \sigma_n^2 \sum_J \alpha_j^{2/2^*} \delta_{p_j}$$

Part V: Reducing to the Case $\lambda(M) < \lambda(S^n)$.

In this final part, we showcase the results of Aubin and Schoen which together fully resolve the Yamabe Problem. The hypotheses of their results perfectly dovetail, and consist of showing that we can find suitable test functions on M (which come from the sphere in Aubin's case, and from the Green's operator of $\Box$ in Schoen's) which have $Q < \lambda(S^n)$. Let's start with Aubin's result:

Theorem: (Aubin) If (M,g) has $\dim M = n \geq 6$, and if M is not locally conformally flat at some $p \in M$, then $\lambda(M) < \lambda(S^n)$.

To prove it, we'll utilize the following extremely useful construction:

Theorem: (Graham) Conformal Normal Coordinates
Let M be a Rm. mfd. and $p \in M$. For each $K \geq 2$, there is a conformal metric g on M such that

$$\det g = 1 + \mathcal{O}(r^K)$$

where $r = |x|$ is the radial distance in g-normal coordinates at p.
If $K \geq 5$, then in these coordinates we also have $S = \mathcal{O}(r^2)$, and $\Delta S = |W|^2/6$ at p.

Remark: The idea here is that normal coordinates are already quite nice for many computations, but we have even more freedom to search for the best normal coordinates in an entire conformal class, since $\lambda(M)$ is invariant.

Proof of Aubin's Theorem: Let $p \in M$ be a point at which $W(p) \neq 0$, and fix conformal normal coordinates at p for $K \geq 2$ as big as we need in the arguments to follow. Suppose $B_{2\rho}(p)$ is a normal ball contained in the coordinate patch, and fix a cutoff function $\eta \in C^\infty(M)$ with

$$\begin{cases} \chi_{B_\rho} \leq \eta \leq \chi_{B_{2\rho}} \\ |\nabla \eta| \lesssim \rho^{-1} \end{cases}$$

Now, let
$$\varphi(r) = \begin{cases} \eta(r) u_\varepsilon(r) & r \leq 2\rho \\ 0 & r \geq 2\rho \end{cases} \in C^\infty(M)$$

where $u_\varepsilon(x) = \left(\dfrac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{n-2}{2}}$ from earlier. We will show that $\varepsilon > 0$ can be chosen so small that $Q(\varphi) < \lambda(S^n)$.

Note that by the work of Talenti and Aubin, we know that

$$\lambda(S^n) = \frac{\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2}{\|u_\varepsilon\|_{2^*}^2} = n(n-2)\left(\int_{\mathbb{R}^n} u_\varepsilon^{2^*}\right)^{2/n}$$

since we can compute that $-\Delta u_\varepsilon = n(n-2) u_\varepsilon^{2^*-1}$. We now proceed to carefully estimate each part of the quotient $Q(\varphi)$:

Seeing as though φ is radial and in normal coordinates $g_{rr}=1$, we have that

$$\int_M |\nabla\varphi|^2 \, dvol_g = \int_{B_{2\rho}} |\partial_r \varphi|^2 \sqrt{\det g}\, dx \le \int_{B_{2\rho}} |\partial_r \varphi|^2 (1+Cr^k)\, dx$$

$$= \int_{B_\rho} |\nabla u_\varepsilon|^2 dx + C\int_{B_\rho} r^k |\nabla u_\varepsilon|^2 dx$$

$$+ \int_{B_{2\rho}\setminus B_\rho} |\nabla(\eta u_\varepsilon)|^2 (1+Cr^k)\, dx$$

Computing directly from the definition of u_ε shows that

$$|\partial_r u_\varepsilon| \le (n-2)\varepsilon^{(n-2)/2} r^{1-n}$$

and so it follows immediately that the last two integrals are $O(\varepsilon^{n-2})$. Next, integrate in parts in the first integral, using $-\Delta u_\varepsilon = n(n-2)u_\varepsilon^{2^*-1}$:

$$\int_{B_\rho} |\nabla u_\varepsilon|^2 = n(n-2)\int_{B_\rho} u_\varepsilon^{2^*} + \int_{\partial B_\rho} u_\varepsilon \partial_r u_\varepsilon$$

$$< n(n-2)\int_{B_\rho} u_\varepsilon^{2^*}$$

as $\partial_r u_\varepsilon < 0$. Thus, since $\frac{2}{n} + \frac{2}{2^*} = 1$,

$$\int_{B_\rho} |\nabla u_\varepsilon|^2 < n(n-2)\left(\int_{B_\rho} u_\varepsilon^{2^*}\right)^{\frac{2}{n}} \left(\int_{B_\rho} u_\varepsilon^{2^*}\right)^{2/2^*}$$

$$= \lambda(S^n)\left(\int_{B_\rho} u_\varepsilon^{2^*}\right)^{2/n}$$

Altogether, $\displaystyle\int_M |\nabla\varphi|^2 dvol_g \le \lambda(S^n)\left(\int_{B_\rho} u_\varepsilon^{2^*}\right)^{2/n} + C\varepsilon^{n-2}$.

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Next, observe that

$$\int_M \varphi^{2^*} dvol_g = \int_{B_\rho} u_\varepsilon^{2^*} \sqrt{\det g}\, dx + \int_{B_{2\rho}\setminus B_\rho} (\eta u_\varepsilon)^{2^*} \sqrt{\det g}\, dx$$

$$\ge \int_{B_\rho} u_\varepsilon^{2^*} dx - C\int_{B_\rho} r^k u_\varepsilon^{2^*} dx - \int_{B_{2\rho}\setminus B_\rho} u_\varepsilon^{2^*} (1+Cr^k)\, dx$$

$$\ge \int_{B_\rho} u_\varepsilon^{2^*} dx - C\varepsilon^n.$$

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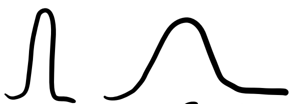
Lastly, we estimate the scalar curvature term. Choosing at least $k \geq 5$ ensures that $S = \mathcal{O}(r^2)$ and $\Delta S(p) = -|W(p)|^2/6$. Since at p we have $\Gamma_{ij}^{k}(p) = 0$ and $\partial_k g_{ij}(p) = 0$, it follows that $S(p) = S_{:i}(p) = 0$ $\forall i$. Thus

$$S = \frac{1}{2} S_{:ij}(p) x^i x^j + \mathcal{O}(r^3)$$

and we obtain

$$\int_M S \varphi^2 \, dvol_g \leq (1 + Cr^k) \int_{B_{2\rho}} \left(\frac{1}{2} S_{:ij}(p) x^i x^j + \mathcal{O}(r^3) \right) \eta^2 u_\varepsilon^2 \, dx$$

$$\leq (1 + Cr^k) \left\{ \frac{1}{2} \int_{B_{2\rho}} S_{:ij}(p) x^i x^j \eta^2 u_\varepsilon^2 \, dx + C \int_{B_{2\rho}} \eta^2 u_\varepsilon^2 r^3 \, dx \right\}$$



$\varepsilon \ll 1$ $\quad$ $\varepsilon > 1$

$r = |x|$

$$= (1 + Cr^k) \left\{ \frac{1}{2} \int_0^{2\rho} \eta^2 u_\varepsilon^2 \int_{|x| = r} S_{:ij}(p) x^i x^j \, d\mathcal{H}^{n-1} \, dr + C \int_0^{2\rho} \eta^2 u_\varepsilon^2 \int_{|x|=r} r^{n+2} \, d\mathcal{H}^{n-1} \, dr \right\}$$

$$= (1 + Cr^k) \left\{ \frac{\omega_{n-1}}{n} \Delta S(p) \int_0^{2\rho} \eta^2 u_\varepsilon^2 r^{n+1} \, dr + C \omega_{n-1} \int_0^{2\rho} \eta^2 u_\varepsilon^2 r^{n+2} \, dr \right\}$$

$$\left(\int_{|x| = r} x^i x^j \, d\mathcal{H}^{n-1} = \frac{\omega_{n-1}}{n} r^{n+1} \delta_{ij} \right)$$

Technical lemma: Let $k > -n$. As $\varepsilon \searrow 0$,

$$I(\varepsilon) := \int_0^{2\rho} u_\varepsilon^2 r^{n+k-1} \, dr$$

satisfies

$$I(\varepsilon) = \begin{cases} \mathcal{O}(\varepsilon^{k+2}) & n > k+4 \\ \mathcal{O}(\varepsilon^{k+2} |\log \varepsilon|) & n = k+4 \\ \mathcal{O}(\varepsilon^{n-2}) & n < k+4 \end{cases}$$

Applying this to the first integral, we have

$$\frac{\omega_{n-1}}{n} \Delta S(p) \int_0^{2\rho} \eta^2 u_\varepsilon^2 r^{n+1} \, dr \leq -C |W(p)|^2 \cdot \begin{cases} \varepsilon^4 |\log \varepsilon| & \text{if } n = 6 \\ \varepsilon^4 & \text{if } n \geq 7 \end{cases}$$

and in the second,

$$C \omega_{n-1} \int_0^{2\rho} \eta^2 u_\varepsilon^2 r^{n+2} \, dr = \begin{cases} \mathcal{O}(\varepsilon^5) & n \neq 7 \\ \mathcal{O}(\varepsilon^5 |\log \varepsilon|) = \mathcal{O}(\varepsilon^5) \end{cases}$$

Thus,

$$\int_M S \varphi^2 \, dvol_g \leq \begin{cases} -C|W(p)|^2 \varepsilon^4 |\log \varepsilon| + \mathcal{O}(\varepsilon^5) & n = 6 \\ -C|W(p)|^2 \varepsilon^4 + \mathcal{O}(\varepsilon^5) & n \geq 7 \end{cases}$$

Altogether, we conclude that

$$Q(\varphi) = \frac{\int c_n |\nabla\varphi|^2 \, dvol_g + \int S\varphi^2 \, dvol_g}{\left(\int \varphi^{2^*} dvol_g\right)^{2/2^*}}$$

$$\leq \Bigg[\lambda(S^n)\left(\int_{B_p} u_\varepsilon^{2^*}\right)^{2/n} + C\varepsilon^{n-2} + (1+C_r k)\begin{cases} -c|w(p)|^2\varepsilon^4 |\log\varepsilon| + \mathcal{O}(\varepsilon^5) & n=6 \\ -c|w(p)|^2\varepsilon^4 + \mathcal{O}(\varepsilon^5) & n\geq 7 \end{cases}\Bigg]$$

$$\cdot \left(\int_{B_p} u_\varepsilon^{2^*} dx\right)^{2/2^*}(1+\mathcal{O}(\varepsilon^5))$$

$$= \begin{cases} \lambda(S^n) - c|w(p)|^2\varepsilon^4|\log\varepsilon| + \mathcal{O}(\varepsilon^5) & n=6 \\ \lambda(S^n) - c|w(p)|^2\varepsilon^4 + \mathcal{O}(\varepsilon^5) & n\geq 7 \end{cases}$$

Since $|w(p)| > 0$, for small enough $\varepsilon > 0$ we obtain our desired result.

That just leaves us wondering about manifolds which have dimensions 3, 4, or 5, or which are loc. conformally flat. Schoen was able to dispense with all of these cases in one result. To begin, we need to return to the topic of stereographic projections.

As before, we'll set $\Psi: S^n \setminus \{N\} \to \mathbb{R}^n$ as the standard stereographic projection but this time we focus on the conformal factor G on $S^n\setminus\{N\}$ defined by

$$\hat{g} := \Psi^\# g_{std} = G^{p-2} g_0 .$$

Since $(\mathbb{R}^n, g_{std})$ has vanishing scalar curvature,

$$0 = \Box_{g_0} G = -c_n \Delta_{g_0} G + n(n-1) G \qquad \text{on } S^n\setminus\{N\}.$$

In fact, it can be shown that this conformal factor G is the Green's operator for $\Box_{g_0}$ at N! That is,

$$\Box_{g_0} G = \delta_N \qquad \text{on } S^n.$$

Now, here is Schoen's idea, which reverses the above sequence of observations. On a closed Rm. mfd (M,g), we can prove the existence of the Green's function G to $\Box_g$. If it were known to be positive, then we could use it as a conformal factor, by fixing $p \in M$ and setting

$$\hat{g} := G_p^{-2} g \qquad \text{on } \hat{M} := M \setminus \{p\}.$$

This gives a map $\sigma : (\hat{M}, \hat{g}) \to (M, g)$ which we call the "stereographic projection of M from p." This is all done so that, like in the Euclidean case,

$$S_{\hat{g}} = 0 \quad \text{on } \hat{M}.$$

Now, if $\lambda(M) \geq 0$ then one can show that $\Box_g$ has $G > 0$. Since the Yamabe Problem is easily solved if $\lambda(M) \leq 0$, we can assume from now on that $G > 0$. Fix $p \in M$, g a conformal metric realizing Graham's conformal normal coordinates $\{x^i\}$.

Theorem: Suppose (M^n, g) as above has $n = 3, 4, 5$, or is conformally flat at p. Then $\exists C$ s.t.

$$G = r^{2-n} + C + O''(r) \qquad \text{as } r \searrow 0$$

Remark: $f = O^k(r^m)$ iff $\partial^\alpha f \in O(r^{m-|\alpha|})$ for any $|\alpha| \leq k$.

Just one interesting and useful consequence of this theorem is that $\hat{M}$ is asymptotically flat at ∞. We recall:

Def: A Riem. mfd (N, h) is asymptotically flat to order $\tau > 0$ if there is a decomposition $N = N_0 \cup N_\infty$ such that N_0 is compact, N_∞ is diffeomorphic to $\mathbb{R}^n \setminus B_r$ for some $r > 0$, and

$$g_{ij} = \delta_{ij} + O''(\rho^{-\tau}) \qquad \text{as } \rho \nearrow \infty$$

Here $\rho = \mathrm{dist}(-, p)$.

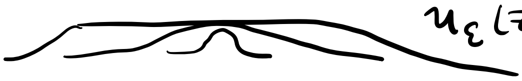
Using the expansion of G, we can give an expansion of $\hat{g}$ in "inverted normal coordinates" $z^i := r^{-2} x^i$ which throw p to ∞. Indeed, on the normal ball $U \setminus \{p\}$ we let $|z| = \rho$ and find that (recall $\hat{g} = G_p^{-2} g$)

$$\hat{g}_{ij}(z) = \left(1 + C\rho^{2-n} + O''(\rho^{1-n})\right)^{p-2} \left(\delta_{ij} + O''(\rho^2)\right)$$

and so we see that $(\hat{M}, \hat{g})$ is asymptotically flat to order

$$\tau = \begin{cases} 1 & n = 3 \\ 2 & n = 4, 5 \\ n-2 & M \text{ is loc. conformally flat.} \end{cases}$$

Now let's proceed to the construction of our test function, using the understanding we've developed about how the "far-reaches" of $(\hat{M}, \hat{g})$ behave. To start, set



$$u_\varepsilon(z) := \begin{cases} u_\varepsilon(z) & \text{if } \rho = |z| \geq R \\ u_\varepsilon(R) & \text{if } \rho = |z| \leq R \end{cases}$$

where again $u_\varepsilon(z) = \left(\frac{\varepsilon}{\varepsilon^2 + |z|^2}\right)^{\frac{n-2}{2}}$ is a dilated Sobolev-extremal functional on $\mathbb{R}^n$. The game now is to take $\varepsilon \gg 1$ to spread u_ε out, and track what happens to $Q_{\hat{g}}(u_\varepsilon)$.

Seeing as though u_ε is radial, as $\varepsilon \nearrow \infty$ we should expect that $Q_{\hat{g}}(u_\varepsilon)$ will depend heavily on what $\hat{g}$ looks like on very large spheres. To this end, we define

$$h(\rho) = \frac{1}{n\omega_n \rho^{n-1}} \int_{S_\rho(p)} d\sigma_{\hat{g}} = \frac{vol_{\hat{g}}(S_\rho)}{vol_{g_{std}}(S_\rho)}$$

Using the metric expansion obtained from the Green's function expansion, we discover that

$$h(\rho) = 1 + \left(\frac{\nu}{k}\right)\rho^{-k} + \mathcal{O}''(\rho^{-k-1}) \qquad (k \text{ a dimensional quantity})$$

(in the cases where M is loc. conf. flat or dim $3,4,5$). ν is called the distortion coefficient of $\hat{g}$, and it is what ties the Yamabe Problem to general relativity. But first, we notice that this expansion enables us to obtain an estimate for $Q_{\hat{g}}$: $\exists\, C > 0$ s.t.

$$Q_{\hat{g}}(u_\varepsilon) \leq \lambda(S^n) - C\nu\varepsilon^k + \mathcal{O}(\varepsilon^{-k-1}) \qquad \text{as } \varepsilon \nearrow \infty$$

Thus, knowing that $\nu > 0$ would complete the argument! The remarkable connection which we need is the following result proven by Schoen and Yau, and the fact that with our hypotheses on M, $\nu = 2m(\hat{g})$:

The Positive Mass Theorem: Let (N^n, g), $n \geq 3$, be asymptotically flat to order $\tau > (n-2)/2$ with $S_g \geq 0$. Then $m(\hat{g}) \geq 0$, with equality iff (N, g) is isometric to $(\mathbb{R}^n, g_{std})$.

The Conclusion to the Yamabe Problem is thus as follows:

If $(\hat{M}, \hat{g})$ is isometric to $(\mathbb{R}^n, g_{std})$, then it is certainly conformal to (S^n, g_0) and we are done. Otherwise, our stereographic projection argument ensures us that $S_{\hat{g}} = 0$, so the PMT yields $m(\hat{g}) = \frac{1}{2}\nu > 0$. Taking $\varepsilon \gg 1$ allows us to conclude at last that

$$\lambda(M) < \lambda(S^n).$$